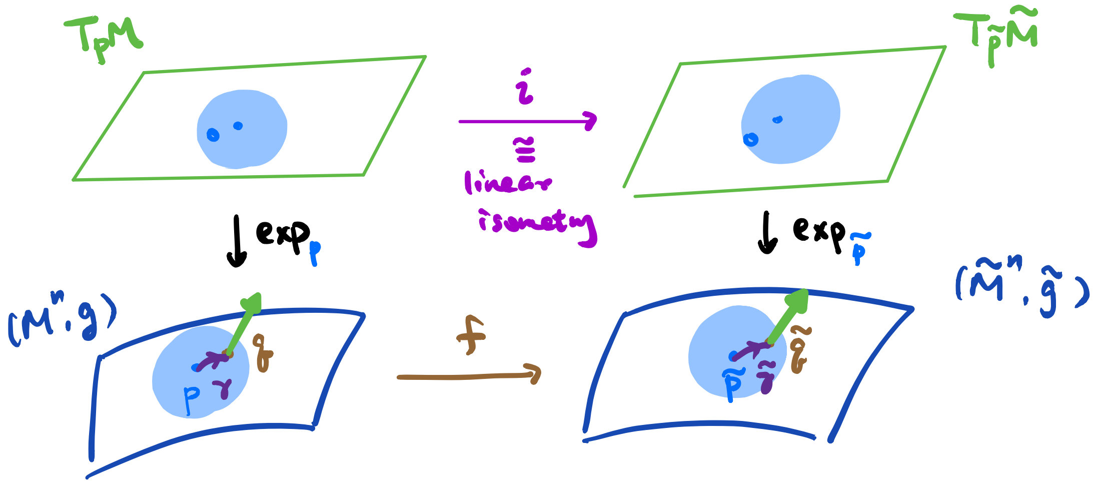
MATH 5061 Lecture 13 (Apr 14)

Last time: Want to "classify" spaces of constant $\overset{\text{sectional}}{\curlyvee}$ curvature

Cartan-Ambrose Lemma: Roughly speaking, the Riemann curvature tensor essentially the Riemannian metric locally (up to isometry).



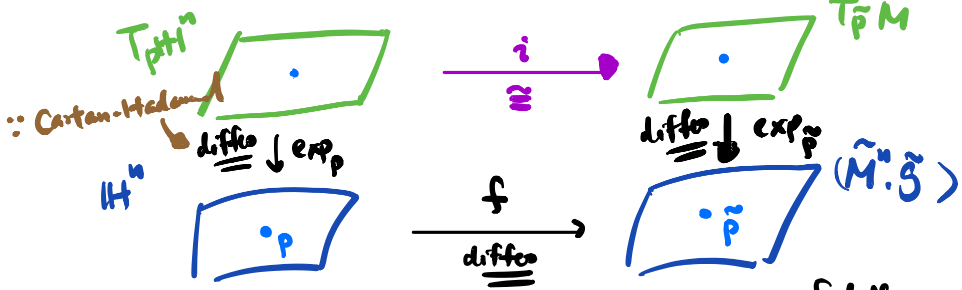
We can now prove:

Cartan's Thm: Let (M^n, g) be a complete Riem. manifold. Suppose (M^n, g) has constant sectional curvature $K_0 \in \{1, 0, -1\}$. THEN, the universal cover $(\tilde M^n, \tilde g)$ is isometric to either

$\mathbb{S}^n$ $(K_0 = 1)$, $\mathbb{R}^n$ $(K_0 = 0)$ or $\mathbb{H}^n$ $(K_0 = -1)$.

"Proof": Case 1: $K_0 = -1$ or $K_0 = 0$

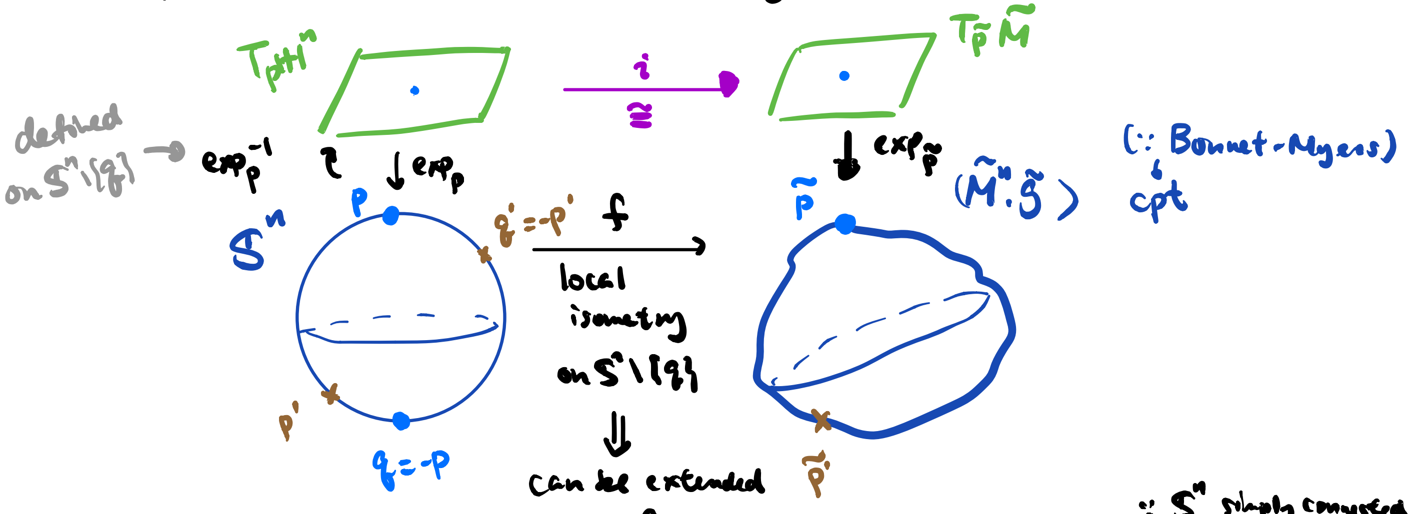
Fix $p \in \mathbb{H}^n$, $\tilde p \in \tilde M$ and isometry $i : T_p\mathbb{H}^n \to T_{\tilde p}\tilde M$



Cartan-Ambrose Lemma $\Rightarrow$ f local isometry $\xRightarrow{f \text{ diffeo}}$ f global isometry

Case 2: $K_0 = 1$

Fix $p \in \mathbb{H}^n$, $\tilde{p} \in \tilde{M}$ and isometry $i : T_p\mathbb{H}^n \to T_{\tilde{p}}\tilde{M}$



$\Downarrow$

can be extended across q

∵ S^n simply connected

remains local isometry ⇒ covering map ⇒ f diffeo.

□

Comparison theorems in Riemannian Geometry

Idea: curvature bdd ⇒ geometric bdd

in comparison with spaces of constant curvature $(\mathbb{H}^n, \mathbb{R}^n, S^n)$.

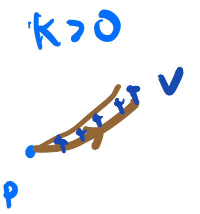
One example is Bonnet-Myers Thm:

(M^n, g) $\mathrm{Ric}^{M^n} \geq \mathrm{Ric}^{S^n}$ ⇒ $\mathrm{diam}(M, g) \leq \mathrm{diam}(S^n)$

complete

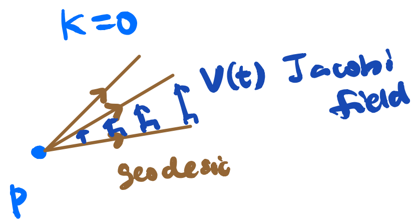
We will discuss another comparison theorem due to Rauch.

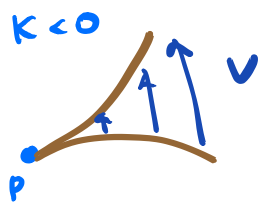
Idea: Curvature ⇝ Spreading of geodesic ⟺ Jacobi field estimates



$|V|$ small

⟺ geodesics converges





$|V|$ big

⟺ geodesic diverges

The following proposition gives a quantitative description of the effect of spreading of geodesics by curvature via the length of Jacobi fields.

Prop: Let $\gamma : [0,a] \to (M^n, g)$ be a geodesic w/. $\gamma(0) = p$, $\gamma'(0) = v$.

For any $w \in T_v(T_pM)$, $|w| = 1$, consider the Jacobi field

$$V(t) := (d\exp_p)_{tv}(tw)$$

Note: $V(0) = 0$, $V'(0) = w$

$V'' + R(\gamma', V)\gamma' = 0$

THEN, we have the following Taylor expansion (near $t=0$)

$$|V(t)|^2 = t^2 - \frac{1}{3} \overset{M}{R}(v,w,v,w)\, t^4 + o(t^4) \quad \text{as } t \to 0$$

Proof: We do Taylor expansion for the function $f(t) := |V(t)|^2$.

$$f(0) = |V(0)|^2 = 0 \qquad (\because V(0) = 0)$$

$$f'(t) = 2\langle V(t), V'(t)\rangle \overset{t=0}{\Rightarrow} f'(0) = 0 \qquad (\because V(0) = 0)$$

$$f''(t) = 2\langle V', V'\rangle + 2\langle V, V''\rangle \overset{\text{at } t=0}{\Longrightarrow} f''(0) = 2|V'(0)|^2 = 2$$

$$= 2|V'|^2 - 2R(\gamma', V, \gamma', V)$$

$$f'''(t) = 4\langle V', \overset{= -R(\gamma',V)\gamma'}{\boxed{V''}}\rangle - 2(\nabla_{\gamma'}R)(\gamma', V, \gamma', V) \overset{\text{at } t=0}{\Longrightarrow} f'''(0) = 0$$

$$- 4R(\gamma', V, \gamma', V')$$

$$f''''(0) = -4R(\gamma', V', \gamma', V') - 4R(\gamma', V', \gamma', V') = -8R(v,w,v,w)$$

□

Cor: When $w \perp v$, then

$$|V(t)| = t - \frac{1}{6} K(\mathrm{span}\{v,w\})\, t^3 + o(t^3)$$

Note: $K \leq \tilde{K}$

$\Rightarrow |V(t)| \geq |\tilde{V}(t)|$

for small $t \approx 0$

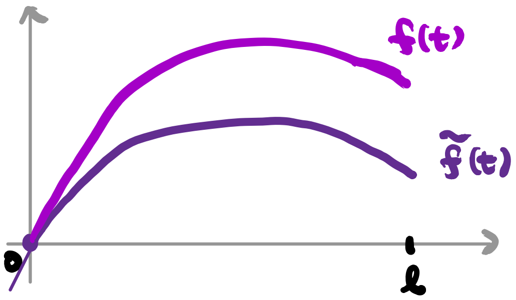
Q: What about for large $t \gg 0$? **A:** Rauch comparison thm.

2D | "Sturm's" Theory for ODE

2D: $V(t) = f(t)\, J\gamma'(t)$; $\tilde{V}(t) = \tilde{f}(t)\, J\tilde{\gamma}'(t)$

$$\begin{cases} f''(t) + K(t) f(t) = 0 , & t \in [0, \ell] \\ f(0) = 0 \end{cases}$$

$$\begin{cases} \tilde{f}''(t) + \tilde{K}(t) \tilde{f}(t) = 0 , & t \in [0, \ell] . \\ \tilde{f}(0) = 0 \end{cases}$$



Suppose $f'(0) = \tilde{f}'(0) > 0$. $\tilde{f}(t) \neq 0 \quad \forall t \in [0, \ell]$.

and $\tilde{K}(t) \geq K(t)$

THEN : $\tilde{f}(t) \leq f(t) \quad \forall t \in [0, \ell]$

Note: We need to be more careful for higher dimensions.

We first establish a useful lemma which says that (normal) Jacobi fields minimize the index form among all (normal) vector fields with the same boundary values.

Index Lemma : Let $\gamma : [0, a] \to M^n$ be a geodesic.

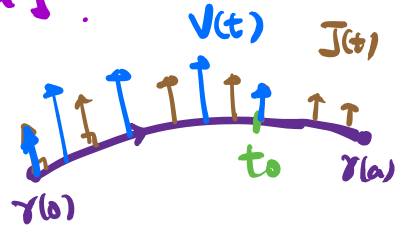
Let $J(t)$ be a normal Jacobi field along γ

Suppose $V(t)$ be any (piecewise smooth) normal vector field along γ

s.t. $J(0) = V(0)$ & $J(t_0) = V(t_0)$ for some $t_0 \in (0, a]$

ASSUME: γ has NO conjugate points on $[0, a]$.

THEN : $\boxed{I_{t_0}(J, J) \leq I_{t_0}(V, V)}$



and "=" holds $\Leftrightarrow$ $J(t) \equiv V(t)$ on $[0, t_0]$.

Here, $I_{t_0}(W,W) := \int_0^{t_0} |W'|^2 - R(\gamma', W, \gamma', W)\, dt$

Proof of Index Lemma:

Let $\mathcal{J} := \{ J(t)$ normal Jacobi fields along γ on $[0,a]$ st. $J(0) = 0 \}$

which is a vector space of $\dim = n-1$. Fix a basis $\{J_1, \dots, J_{n-1}\}$.

$\nexists$ conjugate pt along γ on $[0,a]$ $\Rightarrow \forall t \in [0,a]$, $\{J_1(t), \dots, J_{n-1}(t)\} \overset{\text{basis}}{\subseteq} T_{\gamma(t)}M$

We can write

$$V(t) = \sum_{i=1}^{n-1} \underbrace{f_i(t)}_{\text{piecewise smooth}} J_i(t) \; ; \; J(t) = \sum_{i=1}^{n-1} \underbrace{\alpha_i}_{\text{constants}} J_i(t)$$

Claim: $\langle V', V' \rangle - R(\gamma', V, \gamma', V)$ integrand of $I(V,V)$

$$= \underline{\left\langle \sum_i f_i' J_i, \sum_j f_j' J_j \right\rangle} + \frac{d}{dt}\left(\left\langle \sum_i f_i J_i, \sum_j f_j J_j' \right\rangle \right)$$

Pf of Claim: Recall: $J_i'' = -R(\gamma', J_i)\gamma'$

$$\text{L.H.S.} = \underbrace{\left\langle \sum_i (f_i' J_i + f_i J_i'), \sum_j (f_j' J_j + f_j J_j') \right\rangle}_{\text{expand this}} - \sum_{i,j} f_i f_j \underbrace{R(\gamma', J_i, \gamma', J_j)}_{\substack{\| \\ -\langle J_i'', J_j \rangle}}$$

$$= \sum_{i,j} \Big(\underline{f_i' f_j' \langle J_i, J_j \rangle} + 2 f_i' f_j \langle J_i, J_j' \rangle + f_i f_j \langle J_i', J_j' \rangle + f_i f_j \langle J_i'', J_j \rangle \Big)$$

2nd term of R.H.S. $= \sum_{i,j} \Big(\langle f_i' J_i + f_i J_i', f_j J_j' \rangle + \langle f_i J_i, f_j' J_j' + f_j J_j'' \rangle \Big)$

$$= \sum_{i,j} \left(f_i' f_j \langle J_i , J_j' \rangle + f_i f_j \langle J_i' , J_j' \rangle + f_i f_j' \langle J_i , J_j' \rangle + f_i f_j \langle J_i , J_j'' \rangle \right)$$

Need: $\langle J_i' , J_j \rangle = \langle J_i , J_j' \rangle \quad \forall t \in [0,a]$

Reason: Let $h(t) := \langle J_i' , J_j \rangle - \langle J_i , J_j' \rangle$.

Note $h(0) = 0$ since $J_i(0) = 0 \ \forall i$.

AND: $h'(t) = \langle J_i'' , J_j \rangle + \cancel{\langle J_i' , J_j' \rangle} - \cancel{\langle J_i' , J_j' \rangle} - \langle J_i , J_j'' \rangle$

$$= R(\gamma', J_i, \gamma', J_j) - R(\gamma', J_j, \gamma', J_i) \equiv 0$$

Apply the claim to $V(t)$ and $J(t)$.

$$I_{t_0}(V,V) = \int_0^{t_0} \overbrace{\langle \sum_i f_i' J_i , \sum_j f_j' J_j \rangle}^{= \|\sum_i f_i' J_i\|^2 \geq 0} dt + \underline{\langle \sum_i f_i J_i , \sum_j f_j J_j' \rangle (t_0)}$$

$$I_{t_0}(J,J) = \underline{\langle \sum_i \alpha_i J_i , \sum_j \alpha_j J_j' \rangle (t_0)}$$

$=$

$\because V(t_0) = J(t_0)$

$\Leftrightarrow f_i(t_0) = \alpha_i$

∎

Rauch comparison theorem

Let Riem. mfd (M^n, g)

geodesics $\gamma : [0,a] \to M$

Jacobi fields (normal) J

ASSUME

$|\gamma'| = |\tilde{\gamma}'|$

$J(0) = \tilde{J}(0) = 0$

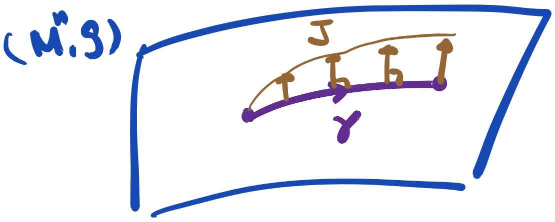
$|J'(0)| = |\tilde{J}'(0)|$

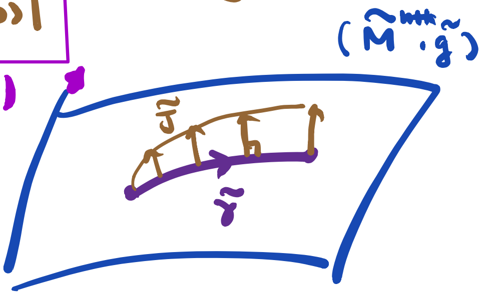
(normalization)

$(\tilde{M}^{n+k}, \tilde{g})$

$\tilde{\gamma} : [0,a] \to \tilde{M}$

$\tilde{J}$





ASSUME: (i) $\nexists$ conjugate pts along $\tilde{\gamma}$ on $[0,a]$

(ii) $K_{\gamma(t)}\left(\operatorname{span}\{\gamma'(t), x\}\right) \leq \tilde{K}_{\tilde{\gamma}(t)}\left(\operatorname{span}\{\tilde{\gamma}'(t), \tilde{x}\}\right)$

$\forall t \in [0,a]$, $\forall\ x \in T_{\gamma(t)}M$, $\tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$

THEN: $|\tilde{J}(t)| \leq |J(t)|$ for ALL $t \in [0,a]$.